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Rigidity and smoothness of motion.

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Abstract. Many theories of structure from motion divide the process into two parts which are solved using different assumptions. Smoothness of the velocity field is often assumed to solve the motion correspondence problem and then rigidity is typically used to recover the 3D structure. We prove results showing that, in a statistical sense, smoothness of the velocity field follows from rigidity of the motion.

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RIGIDITY AND SMOOTHNESS OF MOTION

The smoothness assumption in measuring visual motion

The problem of measuring visual motion is, in many situations, underconstrained. That is, the information in the changing image is insufficent to determine the motion uniquely. This indeterminateness is often referred to as the "aperture problem" (for example Marr and Ullman, 1982). An additional constraint is therefore required for resolving the ambiguity and determining the velocity field uniquely. An important constraint that has been proposed for solving the problem is a smoothness constraint. (Fennema and Thompson 1979, Horn and Schunck 1981, Hildreth 1984, Nakayama and Silverman 1986). When applied to the problem of measuring the velocity of image contours, this smoothness assumption has been formulated by Hildreth in the following manner. Of all the velocity fields that are consistent with the transforming image, select the velocity field that minimizes the overall variation. That is, if v denotes the velocity at a point, and s the arc length along the curve, the preferred velocity field is the one that minimizes the integral:

$$\int \| \frac{dv}{ds} \|^2 \, ds$$

along the curve (the principle of least variation). It has been shown that this method of determining the velocity field gives good results under a wide

range of conditions, and it seems to correspond in many cases to the velocity field perceived by human observers (Hildreth 1986, Nakayama and Silverman 1986).

Justifying the smoothness assumption.

The main rationale raised in support of the smoothness assumption is that the velocity field induced by smooth contours in motion is expected to be smooth. This argument is insufficent, however, for justifying the principle of least variation. The smoothness assumption implies that if the object is known to be smooth, the velocity field should not contain discontinuities. It does not imply, however, that "the smoother the better". Consider, for instance, two points in the image separated by a small distance, and moving with image velocities V_1 and V_2. One may argue that if they lie on the same surface, extremely high values of $||V_1 - V_2||$ are unlikely, because of some physical limitations on the motion of objects in space. But why should one assume that a relative velocity of, say, 0.2 deg/sec is less likely than 0.1 deg/sec? Such a preference is assumed by the least variation principle (that favors velocity fields in which the difference in velocity between neighboring points is as small as possible), but cannot be defended only on the basis of a general smoothness argument. Clearly, a stronger constraint than just the general smoothness of

surfaces is required.

Another general property that seems to provide a useful constraint in the analysis of visual motion is rigidity. Computational studies have shown that the 3D stucture of rigid and quasi-rigid objects can be recovered by looking for the most rigid interpretation possible of the changing image (Ullman 1979, 1984).

In this paper we argue that local rigidity of the object and the principle of least variation in the velocity field are related. To investigate this we consider a rod moving in three-dimensional space. The rod is allowed to move in a semi-rigid manner: it can rotate and translate freely and is also allowed a certain expansion. We calculate the relative velocity between the two endpoints of the rod, as projected on the image plane. The calculations show that under a wide range of conditions this velocity distribution is peaked at zero velocity and decreases monotonically at higher velocities. As a result, the projected velocity field of points linked rigidly together is likely to be consistent with the principle of least variation in the velocity field. There are two factors contributing to this strong bias towards small differential velocities in the projected velocity field: (i) the rigidity of the link between the two points and (ii) the effect of projection from 3-D to the 2-D image plane.

If we imagine an object being made up of a collection of semi-rigid rods

this analysis suggests that the projected velocity field is likely to be smooth. We conclude that the principle of least variation is a reasonable constraint that can be justified for the projected velocity field of rigid or locally rigid objects.

Section 1. The two-dimensional case.

First we consider a rigid rod moving in two dimensions and being projected onto a line. Because it is rigid the rod's velocity can be split into rotational and translational components. For our purposes the translational component can be ignored as it will not contribute to the derivative of the velocity field on the line.

Let the projected length of the rod be R and the real length be r, the angle to the vertical be θ and the angular velocity be ω. Then the projected velocity distribution $\Phi(u)$ is

$$\Phi(u) = \int \delta(R - rsin\theta)\delta(u - r\omega\cos\theta)\Phi_r(r)\Phi_\omega(\omega)drd\omega d\theta \qquad (1.1)$$

where δ denotes the Dirac delta function, $\Phi_r(r)$ and $\Phi_\omega(\omega)$ are the distribution of r and ω respectively. $rsin\theta$ and $r\omega\cos\theta$ are the projected lengths and velocity respectively. We assume that rods are equally likely to have any length between 0 and r_{max} and set $\Phi_r(r) = k$ between 0 and r_{max} (with $kr_{max} = 1$).

We can integrate with respect to θ using the first delta function. This gives (see Appendix 1)

$$\Phi(u) = \int_{r=0}^{r=r_{max}} \frac{1}{\sqrt{r^2 - R^2}} \delta(u - \omega\sqrt{r^2 - R^2}) \Phi_\omega(\omega) dr d\omega. \quad (1.2)$$

We integrate with respect to r to obtain

$$\Phi(u) = \int_{\omega=\omega_{min}}^{\infty} \frac{\Phi_\omega(\omega) d\omega}{(R^2\omega^2 + u^2)^{\frac{1}{2}}} \quad (1.3)$$

where $\omega_{min} = u/\sqrt{r_{max}^2 - R^2}$. This lower bound comes from the delta function integration.

Now we examine the behavior of $\Phi(u)$. If we differentiate it with respect to u we get

$$\frac{\partial \Phi}{\partial u} = -\int_{\omega=\omega_{min}}^{\infty} \frac{u\Phi_\omega(\omega)}{(R^2\omega^2 + u^2)^{\frac{3}{2}}} d\omega - \frac{\Phi_\omega(\omega_{min})}{(r_{max}^2 - R^2)^2 (R^2\omega_{min}^2 + u^2)}. \quad (1.4)$$

So, *for any probability distribution* $\Phi_\omega(\omega)$ (which by definition must be a positive function) the two terms on the right hand side of (1.4) must be negative. $\Phi(u)$ must decrease strictly monotonically with u and has a unique maximum at $u = 0$.

Thus whatever the distribution of the rotation the most likely projected velocity is zero. This result is similar to that obtained by Ullman (1979) for the motion of dots. It was shown that if the motion of a random dot is

described by an isotropic probability distribution function in three dimensions then the probability distribution for the projected two-dimensional motion is peaked at 0.

Note that our result is independent of whether we choose an upper bound for r. This can easily be seen by setting $r_{max} \mapsto \infty$ and $\omega \mapsto 0$ in (1.4). In fact having an upper bound for r makes $\partial\Phi/\partial u$ more negative and makes $\Phi(u)$ decay faster. Henceforth we assume for simplicity that there is no upper bound for r.

To see the connection with the smoothness assumption observe that in the limit Hildreth's smoothness measure can be expressed as

$$\int \|\frac{dv}{ds}\|^2 \, ds = \sum \left(\frac{u}{R}\right)^2 \tag{1.5}$$

where the sum can be taken over a set of rigid rotating rods. The results above show that for each rod considered independently the probability distribution of the u is peaked at zero. Thus, with this independence assumption, we argue that the most likely distribution of the whole contour is the one that minimizes (1.5). We will relax the independence assumption in section 4.

Section 2. The three-dimensional case.

We now extend the analysis to a rod moving in 3-space. Consider a rod of length r projected into the image plane, which has unit normal vector $\vec{k}$.

The rod's direction in 3-D space is denoted by $\vec{r}$ which is a unit vector in the direction $\vec{r}$. The rod is projected into a vector $\vec{r}_p$ where

$$\vec{r}_p = \vec{r} - (\vec{r} \cdot \vec{k})\vec{k}. \qquad (2.1)$$

The rod's rotation is described by a vector $\vec{\omega}$, where ω is the frequency and $\hat{\omega}$ is the axis of rotation. The velocity $\vec{v}$ is given by

$$\vec{v} = \vec{\omega} \times \vec{r}. \qquad (2.2)$$

The projected velocity field $\vec{v}_p$ is given by

$$\vec{v}_p = \vec{v} - (\vec{v} \cdot \vec{k})\vec{k}. \qquad (2.3)$$

We consider rods with projected length R and projected velocity $\vec{U}$. We are interested in the projected velocity distribution $\Phi(\vec{U})$. Since the projected velocity distribution is rotationally symmetric we can express this as $U\Phi(U)$ (see Appendix 1). If the rods' length distribution is $\Phi_r(\vec{r})$ and their rotation distribution is $\Phi_\omega(\vec{\omega})$ then the projected velocity distribution is given by

$$\Phi(u) = \int \delta(R - |\vec{r}_p|)\delta(U - |\vec{v}_p|)\Phi_r(\vec{r})\Phi_\omega(\vec{\omega})d\vec{r}d\vec{\omega}. \qquad (2.4)$$

Care is needed in specifying the domain of integration of (2.4). From (2.2) we see that only the component of $\vec{\omega}$ perpendicular to $\vec{r}$ contributes to the

velocity. We must require $\hat{\vec{\omega}} \cdot \hat{\vec{r}} = 0$. We impose this by substituting a delta function into (2.4) and then integrating over all $\vec{\omega}$.

Thus we have

$$U\Phi(U) = \int \delta(R - |\vec{r}_p|)\delta(\vec{\omega} \cdot \vec{r})\delta(U - |\vec{v}_p|)\Phi_\omega(\vec{\omega})d\vec{r}d\vec{\omega}. \qquad (2.5)$$

We assume all directions of the rod are equally likely and the rods are also equally likely to have any length. So we set

$$\Phi(\vec{r}) = 1. \qquad (2.6)$$

We also assume that the rotation is equally likely to be in any direction. Then $\Phi_\omega(\vec{\omega})$ is a function of $|\vec{\omega}| = \omega$ only.

$$U\Phi(U) = \int \delta(R - |\vec{r}_p|)\delta(\vec{\omega} \cdot \vec{r})\delta(U - |\vec{v}_p|)\Phi_\omega(\omega)d\vec{r}d\vec{\omega}. \qquad (2.7)$$

Choose an orthogonal basis $\vec{i}$, $\vec{j}$, $\vec{k}$. Define angles θ, φ, α, β. These angles are illustrated in figure 1. θ is the angle between the unit vector and the $\vec{k}$ axis. φ is the angle between the unit vector projected onto the x, y plane and the x-axis. Similarly for α and β. It follows that

$$\begin{aligned} \vec{r} &= sin\theta\, cos\varphi\vec{i} + sin\theta\, sin\varphi\vec{j} + cos\theta\vec{k} \\ \vec{\omega} &= sin\alpha\, cos\beta\vec{i} + sin\alpha\, sin\beta\vec{j} + cos\alpha\vec{k} \end{aligned} \qquad (2.8)$$

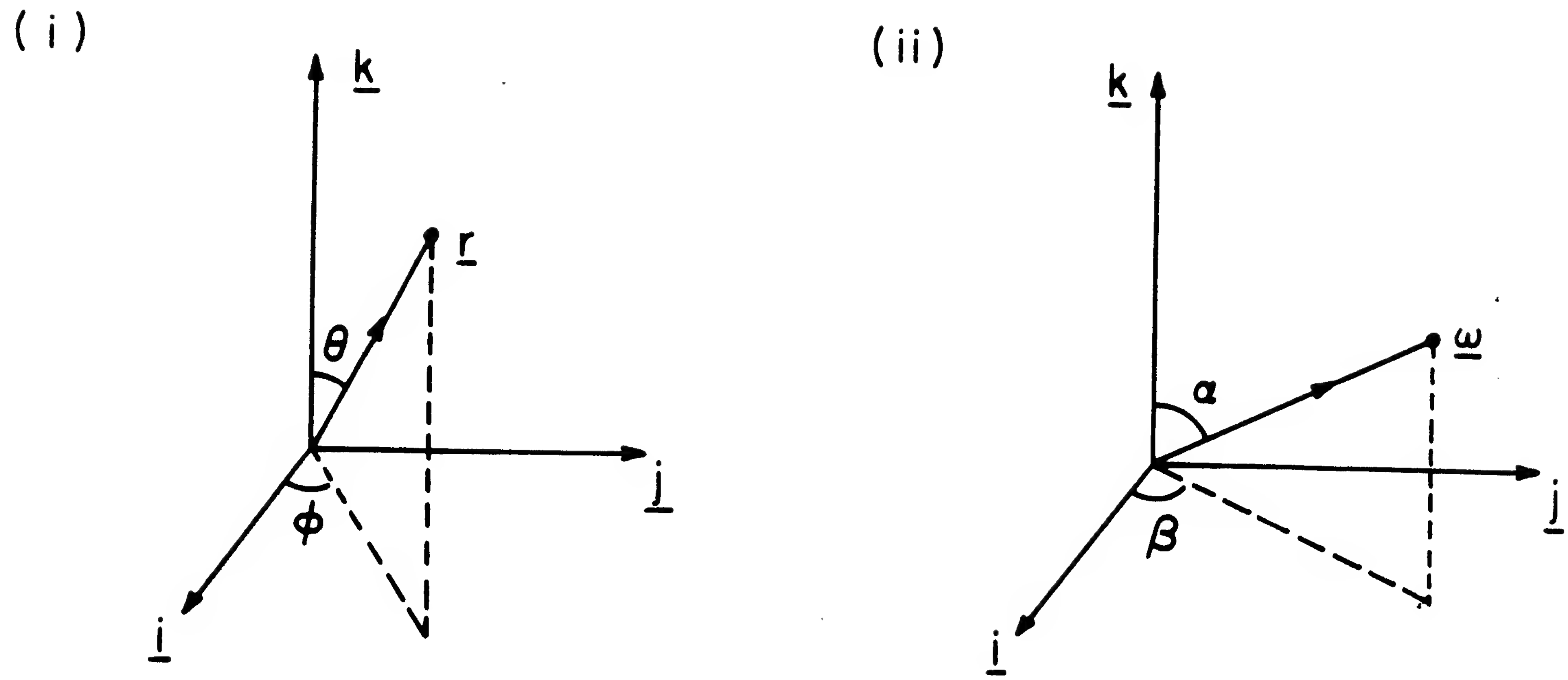

Figure 1 *Definitions of the angles, see text.*

Then

$$d\vec{\omega} = d\omega \; sin\alpha \; d\alpha d\beta$$
$$d\vec{r} = dr \; sin\theta \; d\theta d\varphi \tag{2.9}$$

and

$$|\vec{r}_p| = r sin\theta. \tag{2.10}$$

Define the angle between $\vec{\omega}$ and $\vec{r}$ to be τ by

$$cos\tau = \vec{\omega} \cdot \vec{r}. \tag{2.11}$$

Then from (2.8) we have

$$cos\tau = cos\theta\, cos\alpha + sin\theta\, sin\alpha\, cos(\varphi - \beta). \tag{2.12}$$

Define the angle ρ by

$$cos\rho = \frac{(\vec{\omega} \times \vec{r})}{|\vec{\omega} \times \vec{r}|} \cdot \vec{k}. \tag{2.13}$$

From (2.11) we obtain

$$cos\rho\, sin\tau = (\vec{\omega} \times \vec{r} \cdot \vec{k}). \tag{2.14}$$

From (2.8) we find

$$cos\rho\, sin\tau = sin\theta\, sin\alpha(\varphi - \beta) \tag{2.15}$$

and we calculate

$$|\vec{v}_p| = \omega sin\tau \sin\rho. \tag{2.16}$$

We use (2.10), (2.11) and (2.16) to write the delta functions as

$$\delta(R - |\vec{r}_p|)\delta(\vec{\omega} \cdot \vec{r})\delta(U - |\vec{v}_p|) = \delta(R - r\, sin\theta)\delta(cos\tau)\delta(U - \vec{\omega} r\, sin\tau\, sin\rho). \tag{2.17}$$

We now need to change the variables of integration. We have

$$d\vec{\omega}d\vec{r} = d\omega dr \ sin\alpha \, sin\theta \ d\alpha d\beta d\theta d\varphi. \tag{2.18}$$

The integrand only depends on φ and β as $\varphi - \beta$. We can set $\psi = \varphi - \beta$ and integrate out β. Then we have

$$cos\tau = cos\theta \, cos\alpha + sin\theta \, sin\alpha cos\psi$$

$$cos\rho \, sin\tau = sin\theta \, sin\alpha \, sin\psi \cdot \tag{2.19}$$

$$d\vec{\omega}d\vec{r} = \pi d\omega dr \ sin\alpha \, sin\theta \ d\alpha \, d\psi \, d\theta$$

Now we must change the variables of integration from α, ψ to τ, ρ:

$$d\alpha d\psi = det \begin{pmatrix} \frac{\partial\alpha}{\partial\tau} & \frac{\partial\alpha}{\partial\rho} \\ \frac{\partial\psi}{\partial\tau} & \frac{\partial\psi}{\partial\rho} \end{pmatrix} d\tau d\rho. \tag{2.20}$$

From (2.19) we eliminate the terms containing ψ to obtain

$$cos\alpha = cos\tau \, cos\theta \pm sin\alpha(sin^2\theta - cos^2\rho)^{\frac{1}{2}}. \tag{2.21}$$

Define functions $F(\tau, \rho, \alpha, \psi)$ and$G(\tau, \rho, \alpha, \psi)$ by

$$F(\tau, \rho, \alpha, \psi) = cos\tau - cos\theta \, cos\alpha - sin\theta \, sin\alpha cos\psi$$

$$\cdot \tag{2.22}$$

$$G(\tau, \rho, \alpha, \psi) = cos\rho \, sin\tau - sin\theta \, sin\alpha \, sin\psi$$

It is a standard result of Jacobian transformations (it can be obtained by using formula (2.20) twice, once changing variables from α, ψ to F, G and then changing F, G to τ, ρ) that

$$d\alpha d\psi = det \begin{pmatrix} \frac{\partial F}{\partial \tau} & \frac{\partial F}{\partial \rho} \\ \frac{\partial G}{\partial \tau} & \frac{\partial G}{\partial \rho} \end{pmatrix} / det \begin{pmatrix} \frac{\partial F}{\partial \alpha} & \frac{\partial F}{\partial \psi} \\ \frac{\partial G}{\partial \alpha} & \frac{\partial G}{\partial \psi} \end{pmatrix} d\tau d\rho. \quad (2.23)$$

From (2.22) we obtain

$$det \begin{pmatrix} \frac{\partial F}{\partial \tau} & \frac{\partial F}{\partial \rho} \\ \frac{\partial G}{\partial \tau} & \frac{\partial G}{\partial \rho} \end{pmatrix} = sin^2\tau \, sin\rho \quad (2.24)$$

and

$$det \begin{pmatrix} \frac{\partial F}{\partial \alpha} & \frac{\partial F}{\partial \psi} \\ \frac{\partial G}{\partial \alpha} & \frac{\partial G}{\partial \psi} \end{pmatrix} = sin\theta \, sin\alpha \{ sin\theta \, cos\alpha - cos\psi \, cos\theta \, sin\alpha \}. \quad (2.25)$$

Thus we calculate

$$d\alpha d\psi = \frac{sin^2\tau \, sin\rho}{sin\theta \, sin\alpha \{ sin\theta \, cos\alpha - cos\psi \, cos\theta \, sin\alpha \}} d\tau d\rho. \quad (2.26)$$

We substitute for $cos\psi$ from (2.19) and obtain

$$d\alpha d\psi = \frac{sin^2\tau \, sin\rho}{sin\alpha \{ cos\alpha - cos\theta \, cos\tau \}} d\tau d\rho \quad (2.27)$$

then using (2.21) we find

$$d\alpha d\psi = \frac{sin\tau \, sin\rho}{sin\alpha (sin^2\theta - cos^2\rho)^{\frac{1}{2}}} d\tau d\rho. \quad (2.28)$$

Hence

$$sin\alpha \, sin\theta d\alpha d\psi = \frac{sin\tau \, sin\rho \, sin\theta}{(sin^2\theta - cos^2\rho)^{\frac{1}{2}}} d\tau d\rho. \quad (2.29)$$

We substitute (2.17), (2.18) and (2.29) into (2.7)

$$U\Phi(u) = \int \delta(R - rsin\theta)\delta(cos\tau)\delta(u - \omega r sin\tau\, sin\rho) \cdot \Phi_\omega(\omega)\frac{sin\tau\, sin\rho\, sin\theta}{(sin^2\theta - cos^2\rho)^{\frac{1}{2}}}\, d\tau\, d\rho\, d\theta\, d\omega\, dr \tag{2.30}$$

Now we do the integration with respect to τ to obtain

$$U\Phi(U) = \int \frac{\delta(R - rsin\theta)\delta(U - \omega r sin\rho)}{(sin^2\theta - cos^2\rho)^{\frac{1}{2}}}\Phi_\omega(\omega)\, sin\rho\, sin\theta\, d\rho\, d\theta\, d\omega\, dr. \tag{2.31}$$

We integrate with respect to θ to obtain

$$U\Phi(u) = \int \frac{\delta(U - \omega r sin\rho)\Phi_\omega(\omega) sin\rho (R/r)\, d\rho\, d\omega\, dr}{(r^2 - R^2)^{\frac{1}{2}}(R^2/r^2 - cos^2\rho)^{\frac{1}{2}}}. \tag{2.32}$$

We integrate with respect to ρ and obtain

$$\Phi(U) = \int \frac{R\Phi_\omega(\omega)d\omega dr}{r(r^2 - R^2)^{\frac{1}{2}}(R^2\omega^2 + u^2 - \omega^2 r^2)^{\frac{1}{2}}(\omega^2 r^2 - u^2)^{\frac{1}{2}}}. \tag{2.33}$$

The domain of integration is restricted to the region of the $\omega - r$ plane where the integrand is real (for simplicity we have omitted the bounds of integration in the previous equations). It is specified by

$$\begin{gathered} r \geq R \\ R^2\omega^2 + u^2 \geq \omega^2 r^2 \\ \omega r \geq u. \end{gathered} \tag{2.34}$$

Change the variable of integration from r to X where

$$\omega^2 r^2 = u^2 + X \tag{2.35}$$

then

$$\Phi(u) = \int \frac{R\omega\Phi_\omega(\omega)d\omega dX}{2X^{\frac{1}{2}}(R^2\omega^2 - X)^{\frac{1}{2}}(u^2 + X - \omega^2 R^2)^{\frac{1}{2}}(u^2 + X)} \tag{2.36}$$

with domain of integration

$$X \leq R^2\omega^2$$
$$X \geq 0 \cdot \tag{2.37}$$
$$u^2 + X \geq \omega^2 R^2$$

Define a new variable Y by

$$Y^2 = \omega^2 R^2 - X \tag{2.38}$$

then

$$\Phi(u) = \int \frac{R\omega\Phi_\omega(\omega)d\omega dY}{(\omega^2 R^2 - Y^2)^{\frac{1}{2}}(u^2 - Y^2)^{\frac{1}{2}}(u^2 + \omega^2 R^2 - Y^2)} \tag{2.39}$$

with

$$Y \geq 0$$

$$Y \leq \omega R \tag{2.40}$$

$$Y \leq u.$$

We can further simplify this expression by the substitutions $Y = usin\theta$ and $\omega = (u/R)\Omega$. This yields

$$\Phi(u) = \frac{1}{u^2}\int_0^{\frac{\pi}{2}} d\theta \int_{sin\theta}^{\infty} d\Omega \frac{\Phi_\omega(u\Omega/R)}{(\Omega^2 - sin^2\theta)^{1/2}(\Omega^2 + cos^2\theta)}. \tag{2.41}$$

It is clear from this expression that $\Phi(u)$ will decrease with u if and only if $\Phi_\omega(\omega)$ does not grow faster than ω^2. Thus for all realistic cases $\Phi(u)$ will be a monotonic decreasing function of u.

Section 3. Expansion and Contraction.

We now consider the situation in two dimensions when the rod is allowed to expand and contract as well as rotate. We can write

$$\vec{r} = r\vec{\hat{r}} \tag{3.1}$$

and differentiating gives

$$\dot{\vec{r}} = \dot{r}\vec{\hat{r}} + r\dot{\theta}\vec{\hat{n}} \tag{3.2}$$

where $\vec{\hat{n}}$ is the unit vector perpendicular to $\vec{\hat{r}}$. We define the expansion coefficient s by

$$\dot{r} = rs \tag{3.3}$$

and write (3.2) as

$$\dot{\vec{r}} = rs\vec{r} + r\dot{\theta}\vec{n}. \tag{3.4}$$

The projected velocity is then given by

$$U = r\, s\, sin\theta + r\, \omega\, cos\theta \tag{3.5}$$

where θ is the angle that $\vec{r}$ makes with the normal to the image plane and $\dot{\theta} = \omega$. The projected velocity distribution is given by

$$\Phi(U) = \int \delta(R - rsin\theta)\delta(U - (r\, s\, sin\theta + r\omega cos\theta))\Phi_s(s)\Phi_\omega(\omega)\, dr\, d\omega\, ds\, d\theta \tag{3.6}$$

where, as in Section 1, we have assumed that $\Phi(r) = 1$ so the rods have no preferred lengths. We do the integral of $\Phi(U)$ with respect to θ to obtain

$$\Phi(U) = \int (r^2 - R^2)^{-1/2}\delta(U - sR - \omega(r^2 - R^2)^{1/2})\Phi_s(s)\Phi_\omega(\omega)\, dr\, d\omega\, ds. \tag{3.7}$$

We now integrate with respect to r to find

$$\Phi(U) = \int ((U - sR)^2 + \omega^2 R^2)^{-1/2}\Phi_s(s)\Phi_\omega(\omega) d\omega ds. \tag{3.8}$$

Care must be taken to ensure that this integral is evaluated over the correct limits, those for which the integrand is well defined. For $\omega > 0$ we need $s < \frac{U}{R}$ and for $\omega < 0$ we have $s > \frac{U}{R}$. This domain of integration is shown in figure (2). We can split the integral up into two parts

$$\Phi(U) = \Phi_1(U) + \Phi_2(U) \tag{3.9}$$

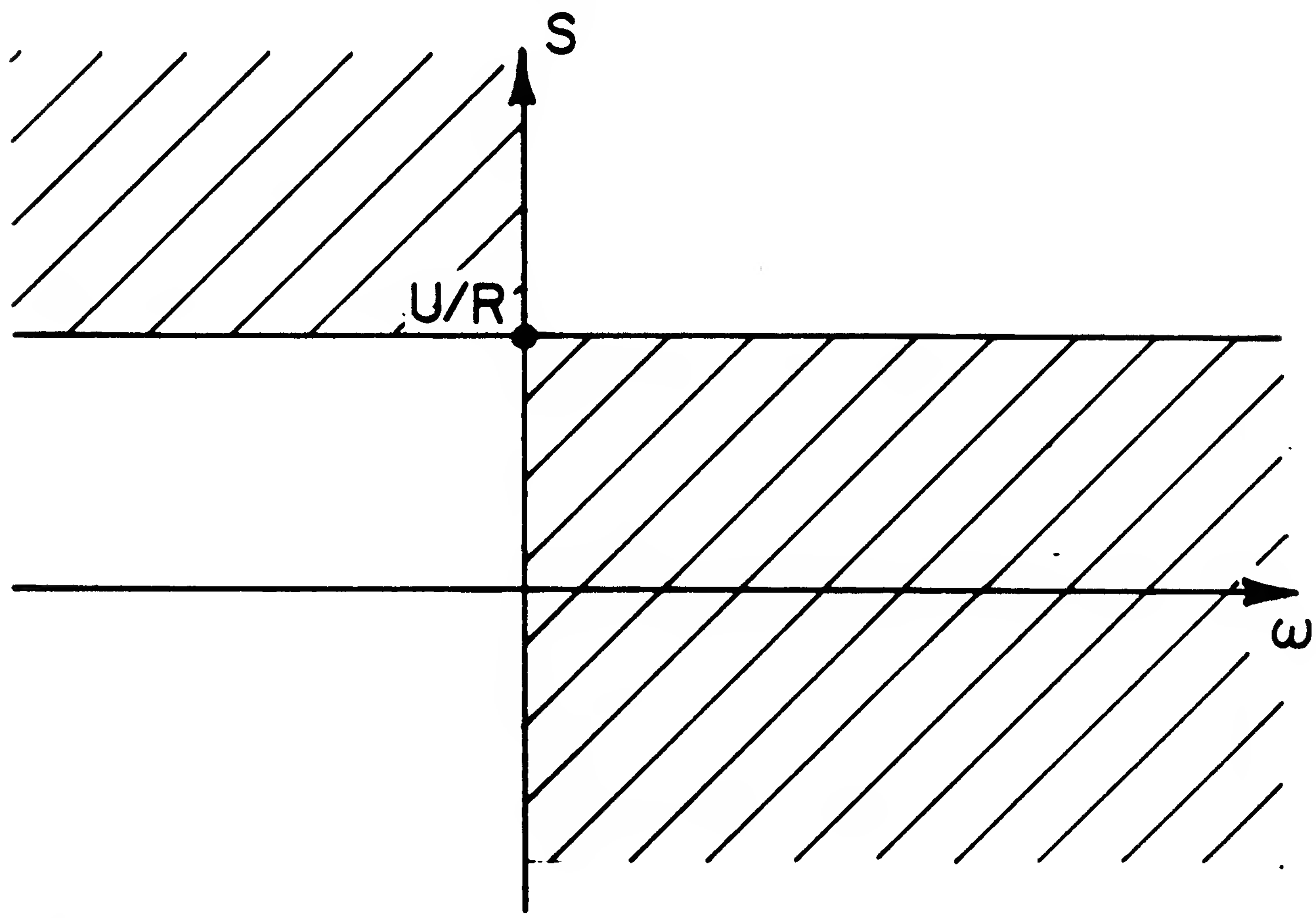

Figure 2 *The domain of integration, see text.*

where

$$\Phi_1(U) = \int_{\omega=0}^{\omega=\infty} d\omega \int_{s=-\infty}^{s=U/R} ((U - sR)^2 + R^2\omega^2)^{-1/2} \Phi_s(s)\Phi_\omega(\omega)ds \tag{3.10}$$

and

$$\Phi_2(U) = \int_{\omega=-\infty}^{\omega=0} d\omega \int_{s=U/R}^{s=\infty} ((U - sR)^2 + R^2\omega^2)^{-1/2}\Phi_s(s)\Phi_\omega(\omega)ds. \quad (3.11)$$

We impose the conditions

$$\Phi_s(s) = \Phi_s(-s) \quad (3.12)$$

and

$$\Phi_\omega(\omega) = \Phi_\omega(-\omega). \quad (3.13)$$

These ensure that contraction is as likely as expansion and that rotation is equally likely to be any direction.

We use (3.12) and (3.13) to rewrite (3.9) as

$$\Phi(U) = \int_{\omega=0}^{\omega=\infty} d\omega \int_{s=-\infty}^{s=U/R} ((U - sR)^2 + R^2\omega^2)^{-1/2}\Phi_s(s)\Phi_\omega(\omega)ds$$
$$+ \int_{\omega=0}^{\omega=\infty} d\omega \int_{s=-\infty}^{s=-U/R} ((U + sR)^2 + R^2\omega^2)^{-1/2}\Phi_s(s)\Phi_\omega(\omega)ds. \quad (3.14a)$$

We now substitute $t = Rs/u$ and $\omega = u\Omega/R$ and obtain

$$\Phi(U) = \frac{u}{R^2}\int_{\Omega=0}^{\Omega=\infty} d\Omega \int_{t=-\infty}^{t=1} ((1-t)^2 + \Omega^2)^{-1/2}\Phi_s(ut/R)\Phi_\omega(u\Omega/R)dt$$
$$+ \frac{u}{R^2}\int_{\Omega=0}^{\Omega=\infty} d\Omega \int_{t=-\infty}^{t=-1} ((1+t)^2 + \Omega^2)^{-1/2}\Phi_s(ut/R)\Phi_\omega(u\Omega/R)dt. \quad (3.14b)$$

Note that if we set $\Phi_s(ut/R) = \delta(ut/R)$ we recover (1.3).

We differentiate (3.14a) to obtain

$$\frac{\partial \Phi}{\partial U} = -\int_{\omega=0}^{\omega=\infty} d\omega \int_{s=-\infty}^{s=U/R} ((U+sR)^2 + R^2\omega^2)^{-3/2}(U+sR)\Phi_s(s)\Phi_\omega(\omega)ds$$
$$-\int_{\omega=0}^{\omega=\infty} d\omega \int_{s=-\infty}^{s=-U/R} ((U-sR)^2 + R^2\omega^2)^{-3/2}(U-sR)\Phi_s(s)\Phi_\omega(\omega)ds. \quad (3.15)$$

We can simplify this expression by setting s to $s - 2U/R$ in the second term on the right hand side. This yields

$$\frac{\partial \Phi}{\partial U} = -\int_{\omega=0}^{\omega=\infty} d\omega \int_{s=-\infty}^{s=-U/R} \frac{1}{((U+sR)^2 + R^2\omega^2)^{3/2}}(sR+U)(\Phi_s(s) - \Phi_s(s+2U/R))\Phi_\omega(\omega)ds.$$
$$(3.16)$$

In the range of integration $(u + sR)$ is always negative. So the integral will be always negative provided $\Phi_s(s) - \Phi_s(s+2u/R)$ is negative for $s \leq -u/R$. If we make s positive, using (3.12), this becomes

$$\Phi_s(s - 2U/R) - \Phi_s(s) \geq 0, \quad (3.17)$$

which is true for all monotonically decreasing functions. Then we have $\frac{\partial \Phi}{\partial U} < 0$ everywhere for any function $\Phi_\omega(\omega)$. Hence the projected velocity distribution will once more be peaked about $U = 0$.

If we relax the conditions (3.17) we are no longer assured of having a monotonically decreasing projected velocity distribution for all $\Phi_\omega(\omega)$. If

$\Phi_s(s)$ is linear in s we obtain a flat distribution

$$\Phi(U) = const. \tag{3.18}$$

If $\Phi_s(s)$ is quadratic in s we find

$$\frac{\partial \Phi}{\partial U} = -2 \int d\omega \frac{\Phi_\omega(\omega)}{\omega} \frac{U^2}{R^4} + \int \int d\omega ds s^2 (s^2 + \omega^2)^{-3/2} \frac{4U}{R^3} \Phi_\omega(\omega) \tag{3.19}$$

which will only be negative for some distributions $\Phi_\omega(\omega)$.

Thus allowing the rod to expand and contract reduces the strength of the result of Section 1. But if we make the reasonable assumption that $\Phi_s(s)$ is at most linear in s the result will still hold.

Section 4. Curves from rods.

The situation becomes more complex when we consider several rods joined together to form a curve. The results above show that for each rod individually the projected velocity distribution is peaked at zero. If the rods are joined, however, these probability distributions are no longer independent. We now sketch an argument suggesting that this loss of independence does not affect the result.

Suppose we have a set of probability distributions $P_i(x_i), \quad i = 1,..N$ which are all peaked at zero. We now define the joint probability distribution $P(\vec{x})$ where $\vec{x} = (x_i, x_2, ..., x_N)$. If we impose no constraints this distribution is given by

$$P(\vec{x})d\vec{x} = \Pi_i P_i(x_i)\Pi_i dx_i. \tag{4.1}$$

Now impose a consistency constraint $\sum_i x_i = 0$. We now have

$$\int P(\vec{x})d\vec{x} = \int \Pi_i P_i(x_i)\delta(\sum_i x_i)\Pi_i dx_i. \tag{4.2}$$

The x_i's can no longer vary independently and the delta function imposes the consistency constraint. We can get integrate out the delta function thereby reducing the number of independent variable to $N-1$. It is convenient to choose these to be $x_i - x_{i+1}$, for $i = 1, ..., N-1$. We can now write (4.2) as

$$\int P(\vec{x})d\vec{x} = \int \Pi_i P_i(x_i - x_{i+1})\Pi_i d(x_i - x_{i+1}). \tag{4.3}$$

We can set $y_i = x_i - x_{i+1}, \quad i = 1, ..., N-1$ and rewrite

$$P(\vec{y}) = P_1(y_1)P_2(y_2)...P_{N-1}(y_{N-1})P_N(-y_1 - y_2 - ... - y_{N-1}). \tag{4.4}$$

It is straightforward to see that the maximum value of $P(\vec{y})$ occurs when the y_i are all zero. Thus P is still peaked at zero even with the constraints.

Section 5. An alternative approach.

The results derived in the last few sections support the assumption (Hildreth 1984) that the velocity field along a contour is smooth. This assumption

is used to solve the aperture problem by prescribing the smoothest possible velocity field consistent with the data.

An alternative approach to the aperture problem was taken by Waxman and Wohn (1986). They showed that if the object was locally planar and moving rigidly then, assuming perspective projection, the local velocity field on the image plane would be locally quadratic in the x, y coordinates. They then defined regions in which these expansions were valid and checked for consistency between these regions. In these regions they could do a least squares best fit to find the coefficients of these quadratic polonomials, and hence the velocity field.

This approach, however, can also be used to justify the smoothest velocity field assumption. The orthographic projection of the motion of a rigid body will obey

$$v_x = A + By + Cz, \tag{5.1a}$$

$$v_y = D + Ex + Fz, \tag{5.1b}$$

where A, B, C, D, E, F are constants and z is the (unknown) depth coordinate. If we also assume the object is planar then z obeys

$$z = px + qy + s, \tag{5.2}$$

where p, q, r are constants. Substituting (5.2) into (5.1) gives linear expressions for v_x and v_y in terms of x and y. So if we assume local planarity and local rigidity (in the style of Waxman and Wohn (1986)) the velocity fields will locally be linear in x and y. The measure of smoothest velocity, $I(\vec{v})$, used by Hildreth is the integral along the curve of the function

$$J(\vec{v}) = \frac{\partial \vec{v}}{\partial s} \cdot \frac{\partial \vec{v}}{\partial s}, \tag{5.3}$$

where s is the arc length. The velocity fields will be locally linear if and only if $J(\vec{v})$ is zero. Thus we can think of the smoothest velocity field approach as a local method of assuming local rigidity and local planarity.

Section 6. Conclusion.

The arguments in the first four sections of this paper suggest that locally rigid, or semi-rigid objects, will tend to project a smooth velocity field in the image. Moreover, assuming random motions and limited expansion or contraction, this field will tend to be as smooth as possible. In the fifth section we noted that rigidity of an object and local smoothness of its surface will also lead to a smooth image motion.

These arguments support the view that maximixing smoothness is a good heuristic to use for motion correspondence and that it is a sensible way to solve the aperture problem.

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Appendix

We first describe the general method for integrating delta functions.

Suppose we have an integral $I(a,b)$

$$I(a,b) = \int_a^b f(x)\delta(x - x_0)dx \qquad (A.1)$$

where $\delta(x)$ is the Dirac delta function, $f(x)$ is an arbitrary function, x_0 an arbitrary point and $b \geq a$. The value of the integral is

$$I(a,b) = f(x_0), \quad if \quad x_0 \epsilon [a,b] \qquad (A.2a)$$

$$I(a,b) = 0, \quad otherwise. \qquad (A.2b)$$

This result can be generalized to integrals of form

$$J(a,b) = \int_a^b f(x)\delta(g(x) - c)dx \qquad (A.3)$$

where $g(x)$ is an arbitrary function and c an arbitrary number. All points x_i with $g(x_i) = c$ will contribute to this interval. Suppose there is only one such point. If there are several we can divide the integral up into regions with only one such point. Consider one such point $x = 0$. The function $g(x)$ can be expanded in a Taylor series about this point

$$g(x) = c + g'(0)(x) + O(x^2). \qquad (A.4)$$

If we change the coordinate to u where $u = g'(0)x$ we can write the integral as

$$J(a,b) = \int_{g'(0)a}^{g'(0)b} f(u/g'(0))\delta(u + O(u^2))\frac{du}{g'(0)}. \qquad (A.5)$$

The value of the integral will depend on the sign of $g'(0)$. If it is negative the bounds of the integral will be reversed and the integral will change sign. Therefore

$$J(a,b) = f(0)\frac{1}{|g'(0)|}, \quad 0\epsilon[a,b] \qquad (A.6a)$$

$$J(a,b) = 0, \quad otherwise. \qquad (A.6b)$$

We now consider the form of the probability distribution function for a rotational symmetric vector in two-dimensional space. Suppose the function is $\Phi_{\vec{\omega}}(\vec{\omega})d\vec{\omega}$. If it is rotationally symmetric (and hence depends only on the modulus ω of $\vec{\omega}$) it can be written

$$\Phi_{\vec{\omega}}(\vec{\omega})d\vec{\omega} = \Phi_{\omega}(\omega)\omega d\omega d\varphi, \tag{A.7}$$

by changing to radial coordinates ω, φ in the ω space. The φ component can be integrated out to give a distribution $\omega\Phi_{\omega}(\omega)d\omega$.

References

Fennema, C.L. and Thompson, W.B. 1979. Velocity determination in scene containing several moving objects,*Comp. Graph. Im. Proc.* *(9)* 301-315

Hildreth, E.C. 1984. *The Measurement of Visual Motion.* Cambridge: MIT Press.

Horn, B.K.P. and Schunck, B.G. 1981. "Determining optical flow." *Art. Intell.* *(17)* 185-203.

Nakayama, K. and Silverman, G.H. 1986. "The aperture problem II: Spatial integration of velocity information along contours." *Submitted for publi-*

cation

Ullman, S. 1979. *The Interpretation of Visual Motion.* Cambridge: MIT Press.

Ullman, S. 1984. "Maximizing rigidity: the incremental recovery of 3-D structure from rigid and non-rigid motion," it Perception (13), 255-274.

Waxman, A.M. and Wohn, K. 1986. "Image flow theory: A framework for 3-D inference from time-varying imagery," In *Advances in Computer Vision.* Eds C.Brown. Erlbaum Publishers.